Some Solution of Important Questions

BUDDHA, GORAKHPUR

Q2. Show that the function defined by $f(z) = \sqrt{|xy|}$ is not analytic (Regular) at the origine (AKTU 2017)

Solution: Given function is

$$f(z) = u(x,y) + iv(x,y) = \sqrt{|xy|}$$

$$\Rightarrow \quad u(x,y) = \sqrt{|xy|} \; ; \quad v(x,y) = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{\text{at }(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{\text{at }(0,0)} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{\text{at }(0,0)} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{\text{at }(0,0)} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

From above we find $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$ at $(0,0)$

Hence Cauchy-Riemann equations are satisfied at origine.

Now $$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|xy|} - 0}{x+iy}$$

If $z \to 0$ along the line $y = mx$ we get

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x(1+im)} = \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

$$f'(0) = \frac{\sqrt{|m|}}{1+im} \quad \text{(depends on 'm')}$$

Since derivative of $f(z)$ at origine depends on m so it is not unique. Therefore $f'(0)$ does not exist.

This confirms that $f(z)$ is not analytic at origine

Q 3. (i) Examine the nature of the function BUDDHA, GORAKHPUR

$$f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$ [AKTU 2016, 2019, 2022]

in the region including the origine

Solution Given function

$$f(z) = u(x,y) + iv(x,y) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$$

$$\Rightarrow u(x,y) = \frac{x^3y^5}{x^4+y^{10}} ; \quad v(x,y) = \frac{x^2y^6}{x^4+y^{10}}$$

$$\left(\frac{\partial u}{\partial x}\right) \text{at } (0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0$$

$$\left(\frac{\partial u}{\partial y}\right) \text{at } (0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y-0} = 0$$

$$\left(\frac{\partial v}{\partial x}\right) \text{at } (0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0$$

$$\left(\frac{\partial v}{\partial y}\right) \text{at } (0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y-0} = 0$$

Thus from above we find

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

Hence C-R. Equations are satisfied at origine

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{\dfrac{x^2y^5(x+iy)}{x^4+y^{10}} - 0}{(x+iy) - 0}$$

$$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2y^5}{x^4+y^{10}}$$

Let $z\to 0$ along ~~line~~ st. line $y = mx$ then

$$f'(0) = \lim_{x\to 0} \frac{x^7 m^5}{x^4+m^{10}x^{10}} = \lim_{x\to 0} \frac{m^5x^2}{1+m^{10}x^6} = 0$$

Further let $z\to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$ This shows that $f'(0)$ is not unique. Hence $f'(0)$ doesnot exist implies $f(z)$ is not analytic at origine although C-R. Equations are satisfied at origine

Q3 (ii)

- If $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$;

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prove that $\dfrac{f(z) - f(0)}{z - 0} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner also that $f(z)$ is not analytic at $z = 0$.

Solution! Given function is

$$f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2} = \frac{x^3 y^2}{x^6 + y^2} - i \frac{x^3 y}{x^6 + y^2}$$

$$\frac{f(z) - f(0)}{z - 0} = \frac{\dfrac{x^3 y(y - ix)}{x^6 + y^2} - 0}{(x + iy) - 0} = \frac{x^3 y (y - ix)}{(x^6 + y^2)(x + iy)}$$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y(y - ix)}{(x^6 + y^2)(x + iy)}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$\lim_{z \to 0} \frac{f(z) - f(0)}{(z - 0)} = \lim_{x \to 0} \frac{x^5 (m - i)}{x^2(x^4 + m^2) \cdot x(1 + im)} = \lim_{x \to 0} \frac{x^2 (m - i)}{(x^4 + m^2)(1 + im)}$$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = 0$$

Hence $\dfrac{f(z) - f(0)}{z - 0} \to 0$ as $z \to 0$ along radius vector $y = mx$.

Let $z \to 0$ along any manner we consider along curve $y = x^3$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{x \to 0} \frac{-i x^3 \cdot x^3}{x^6 + x^6} = \frac{-i}{2}$$

$\Rightarrow \dfrac{f(z) - f(0)}{z - 0}$ does not tend to 0 as $z \to 0$ along curve $y = x^3$

$\therefore f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z - 0}$ doesnot exist.

Hence $f(z)$ is not Analytic at origine $(z = 0)$

Q4 Show that the function $f(z)$ is defined by
$$f(z)=\frac{x^3y^5(x+iy)}{x^6+y^{10}}\ ;\ z\neq 0$$ [AKTU 2023]

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$f(0)=0$ is not analytic at origine even though it satisfies Cauchy-Riemann equation at origine.

Solution: The given function $f(z)=\frac{x^3y^5(x+iy)}{x^6+y^{10}}$

$$\therefore\ f(z)=u(x,y)+iv(x,y)=\frac{x^4y^5}{x^6+y^{10}}+i\frac{x^3y^6}{x^6+y^{10}}$$

gives $u(x,y)=\frac{x^4y^5}{x^6+y^{10}}$; $v(x,y)=\frac{x^3y^6}{x^6+y^{10}}$

$$\left(\frac{\partial u}{\partial x}\right)_{\text{at }(0,0)}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0$$ $\left\{\because f(0)=0,\ u(0,0)=0,\ v(0,0)=0\right.$

$$\left(\frac{\partial u}{\partial y}\right)_{\text{at }(0,0)}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0$$

$$\left(\frac{\partial v}{\partial x}\right)_{\text{at }(0,0)}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0$$

$$\left(\frac{\partial v}{\partial y}\right)_{\text{at }(0,0)}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0$$

From above we find $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$; $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$ at origine

Therefore C-R. Equations are satisfies at origine

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{\frac{x^3y^5(x+iy)}{x^6+y^{10}}-0}{(x+iy)-0}$$

$$f'(0)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y^5}{x^6+y^{10}}$$

Let $z\to 0$ along st. line $y=mx$ we get

$$f'(0)=\lim_{x\to 0}\frac{x^8\cdot m^5}{x^6(1+m^{10}x^4)}=\lim_{x\to 0}\frac{x^2\cdot m^5}{1+m^{10}x^4}=0$$

Let $z\to 0$ along curve $y^5=x^3$ we get

$$f'(0)=\lim_{x\to 0}\frac{x^6}{2x^6}=\frac{1}{2}\Rightarrow f'(0)\text{ does not exists}$$

Hence $f(z)$ is not analytic at origine.

Q5. Show that the function

$$f(z) = \begin{cases} \dfrac{2xy(x+iy)}{x^2+y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$ [AKTU 2016]

The C-R equations are satisfied at origine but derivative of $f(z)$ does not exists at origine.

Solution :- Given function

$$f(z) = u(x,y) + iv(x,y) = \frac{2xy(x+iy)}{x^2+y^2} = \frac{2x^2y}{x^2+y^2} + i\frac{2xy^2}{x^2+y^2}$$

$$\Rightarrow \quad u(x,y) = \frac{2x^2y}{x^2+y^2} \; ; \; v(x,y) = \frac{2xy^2}{x^2+y^2} \; ; \; \because f(0)=0 \Rightarrow u(0,0)=0, \; v(0,0)=0$$

At origine (0,0)

$$\left(\frac{\partial u}{\partial x}\right)_{\text{at }(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{\text{at }(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y-0} = 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{\text{at }(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{\text{at }(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y-0} = 0$$

Hence $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = 0$ C-R. Equations are satisfied at origine,

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\frac{2xy(x+iy)}{x^2+y^2} - 0}{(x+iy)} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy}{x^2+y^2}$$

changing in polar form $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r$

$\therefore x\to 0$ & $y\to 0 \Rightarrow r\to 0$

$$f'(0) = \lim_{r\to 0} \frac{2\cancel{r^2}\sin\theta\cos\theta}{\cancel{r^2}(\cos^2\theta+\sin^2\theta)} = \frac{2\sin\theta\cos\theta}{1}$$

$f'(0) = 2\sin\theta\cos\theta = \sin 2\theta$ depends on θ not unique. Hence derivative of $f(z)$ at origine does not exists.

Q6 Prove that the function $f(z)$ defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \; ; \; z \neq 0$$ [AKTU 2015, 2017]

$$f(0) = 0$$

is continuous and the C-R. equations are satisfied at origin, yet $f'(0)$ does not exist.

Solution: Given function $f(z) = \dfrac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}$; $z \neq 0$

$$f(z) = u(x,y) + iv(x,y) = \left(\frac{x^3-y^3}{x^2+y^2}\right) + i\left(\frac{x^3+y^3}{x^2+y^2}\right) \; ; \; x \neq 0, \; y \neq 0$$

$$\Rightarrow u(x,y) = \frac{x^3-y^3}{x^2+y^2} \; ; \; v(x,y) = \frac{x^3+y^3}{x^2+y^2} \; ; \; x \neq 0, y \neq 0$$

$f(0) = 0 \Rightarrow u(0,0) = 0$ & $v(0,0) = 0$

Since $x \neq 0, y \neq 0 \Rightarrow x^2+y^2 \neq 0$

we know that Any rational function with not zero denominator is continuous for $x \neq 0, y \neq 0$ ie $z \neq 0$

Therefore $u(x,y)$ & $v(x,y)$ are continuous for $x \neq 0, y \neq 0$

Hence $f(z) = u(x,y) + iv(x,y)$ is continuous for $z \neq 0$

Now for continuity of $f(z)$ at $z = 0$

$$\lim_{z \to 0} |f(z) - f(0)| = \lim_{\substack{x \to 0 \\ y \to 0}} \left| \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} - 0 \right|$$

Changing in polar form $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

Since $x \to 0, y \to 0$ & $x^2+y^2 = r^2 \Rightarrow r \to 0$

$$\lim_{z \to 0} |f(z) - f(0)| = \lim_{r \to 0} \left| \frac{r^3\{(\cos^3\theta - \sin^3\theta) + i(\cos^3\theta + \sin^3\theta)\}}{r^2} \right|$$

$$\lim_{z \to 0} |f(z) - f(0)| = \lim_{r \to 0} r \left| (\cos^3\theta - \sin^3\theta) + i(\cos^3\theta + \sin^3\theta) \right|$$

$$= 0 \qquad \text{Hence } \lim_{z \to 0} f(z) = f(0) = 0$$

$\Rightarrow f(z)$ is continuous at origin.

PTO

At origine

$$\left(\frac{\partial u}{\partial x}\right)_{\text{at }(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x-0} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{\text{at }(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = \lim_{y\to 0} \frac{-y-0}{y-0} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{\text{at }(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x-0} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{\text{at }(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \lim_{y\to 0} \frac{y-0}{y-0} = 1$$

from above $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 1$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -1$

Hence C-R equations are satisfied at origine.

Now

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{(z-0)} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2} - 0}{(x+iy) - 0}$$

$$f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Changing in polar foam $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

and $x\to 0, y\to 0 \Rightarrow r\to 0$

$$f'(0) = \lim_{r\to 0} \frac{\cancel{r^3}\left[(\cos^3\theta - \sin^3\theta) + i(\cos^3\theta + \sin^3\theta)\right]}{\cancel{r^3}\,(\cos\theta + i\sin\theta)}$$

$$= \frac{(\cos^3\theta - \sin^3\theta) + i(\cos^3\theta + \sin^3\theta)}{(\cos\theta + i\sin\theta)} = f(\theta) \text{ function of } \theta$$

$f'(0)$ depends on θ not unique.

Hence $f'(0)$ does not exists

$\Rightarrow f(z)$ is not analytic at origine

Q7 Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ is harmonic. Also find the analytic function $f(z) = u(x,y) + iv(x,y)$
[AKTU 2019]

Solution :-

$u(x,y) = x^4 - 6x^2y^2 + y^4 \Rightarrow \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$; $\frac{\partial u}{\partial y} = -12x^2y + 4y^3$

$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$; $\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{12x^2} - \cancel{12y^2} - \cancel{12x^2} + \cancel{12y^2} = 0$$

$u(x,y)$ satisfies Laplace's eqn. Hence $u(x,y)$ is harmonic

By Milne's thomson method

$$f(z) = u + iv = \int \{\phi_1(z,0) - i\,\phi_2(z,0)\}\,dz + C$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{\substack{x=z\\y=0}} = 4z^3 \;;\; \phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{\substack{x=z\\y=0}} = 0$$

$$f(z) = u + iv = \int (4z^3 - i0)\,dz + C = \cancel{4}\frac{z^4}{\cancel{4}} + C$$

$$\boxed{f(z) = z^4 + C}$$ Ans.