BUDDHA SERIES

(Unit Wise Solved Question & Answers)

Course – B.Tech (ASH)

College – Buddha Institute of Technology

(AKTU CODE-525)

Department: Applied Science and Humanities

Subject: Engineering Mathematics II

Q.

Ques: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \cos x$.

Soln. Given diff. Eqn.

$(D^2 - 2D + y) = xe^x \cos x$.

$(D^2 - 2D + 1) y = xe^x \cos x$.

Now, A.E is -

$m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$C.F = (c_1 + c_2 x) e^x$

$P.I = \frac{1}{f(D)} \cdot Q$

$= \frac{1}{D^2 - 2D + 1} \cdot xe^x \cos x$.

$= e^x \cdot \frac{1}{(D+1-1)^2} x \cos x$

$= e^x \cdot \frac{1}{D^2} (x \cos x)$

$$= e^x \cdot \frac{1}{D}\left[\int \underset{I}{x} \cdot \underset{II}{\cos x}\, dx\right]$$

$$= e^x \cdot \frac{1}{D}\left[x\sin x - \int \sin x\, dx\right]$$

$$= e^x \cdot \frac{1}{D}\left[x\sin x + \cos x\right]$$

$$= e^x \left[\int \underset{I}{x}\, \underset{II}{\sin x}\, dx + \int \cos x\, dx\right]$$

$$= e^x \left[-x\cdot\cos x - \int \sin \cos x\, dx + \int \cos x\, dx\right]$$

$$= e^x\left[-x\cos x\right]$$

P.I. $= -e^x x\cos x$

C.S = C.F + P.I

$$C.S = (c_1 + c_2 x)e^x - e^x[x\cos x]$$

Ans

Type-5: Evaluate Particular Integral if -

$$Q(x) = x^m \sin ax \text{ or } x^m \cos ax$$

$$P.I = \frac{1}{F(D)} x^m \cos ax \text{ or } \frac{1}{F(D)} x^m \sin ax$$

$$= \frac{1}{F(D)} x^m \, R.P(e^{iax}) \text{ or } \frac{1}{F(D)} x^m \, I.P(e^{iax})$$

$$= R.P\left[e^{iax} \cdot \frac{1}{F(D+ia)} x^m\right] \text{ or } I.P\left[e^{iax} \frac{1}{F(D+ia)} x^m\right]$$

Ques $(D^2+2D+1)y = x\cos x$

Sol^n:- Given, $(D^2+2D+1)y = x\cos x$ —(*)

Now, A.E is -

$$m^2+2m+1=0$$

$$(m+1)^2$$

$$m = -1, -1$$

$x^m \sin ax$

$\frac{1}{}$ I.P R.P e^{ia} e^{iax}

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$$C.F = (c_1 + c_2 x) e^{-x}$$

$$P.I = \frac{1}{F(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+2D+1} x\cos x$$

$$= \frac{1}{(D+1)^2} \{x \; R.P\, (e^{ix})\}$$

$$= R.P \left[\frac{1}{(D+1)^2} x e^{ix}\right]$$

$$= R.P \left[e^{ix} \frac{1}{(D+i+1)^2} x\right]$$

$$= R.P \left[e^{ix} \frac{1}{D^2+2(1+i)D+(i+1)^2} x\right]$$

$$= R.P \left[e^{ix} \frac{1}{D^2+2(1+i)+2i} x\right]$$

$$= R.P \left[\frac{e^{ix}}{2i} \left\{1+\left(\frac{1+i}{i}D + \frac{D^2}{2i}\right)\right\}^{-1} x\right]$$

$$= R.P \left[\frac{e^{ix}}{2i} \left\{1-\left(\frac{1+iD}{2i} + \frac{D^2}{2i}\right)+ \cdots\right\} x\right]$$

$$= R.P \left[\frac{e^{ix}}{2i} \left\{x - \frac{1+i}{i} + 0\right\}\right]$$

$$= R.P \left[\frac{e^{ix}}{2i} \frac{(ix-1-i)}{i}\right]$$

$$= R.P \left[\frac{e^{ix}}{2} (1+i-ix)\right]$$

$$= R.P \left[\frac{(\cos x + i\sin x)}{2} \{1+i-ix\}\right]$$

$$= \frac{1}{2}\left[\cos x - (1-x)\sin x\right]$$

Solⁿ of eqⁿ (*) is given by

$$C.S = C.F + P.I$$

$$y(x) = (c_1+c_2 x)e^{-x} + \frac{1}{2}\left[\cos x - (1-x)\sin x\right]$$ Ans

Imp.
Que. Find the general solution of $(D^2+a^2)y = \sec ax$ (2017)

Soln. Given that -

$$(D^2+a^2)y = \sec ax$$

Now, A.E is -

$$m^2+a^2=0$$

$$\Rightarrow m = \pm ia$$

$$C.F = (C_1 \cos ax + C_2 \sin ax)$$

$$P.I = \frac{1}{D^2+a^2} \sec ax$$

$$= \frac{1}{(D+ia)(D-ia)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

(i) $\frac{1}{D-ia} \sec ax = e^{iax} \int e^{-iax} \sec ax\, dx$

$$= e^{iax} \int (\cos ax - i \sin ax) \sec ax\, dx$$

$$= e^{iax} \int \left(1 - \frac{i \sin ax}{\cos ax}\right) dx$$

$$= e^{iax} \left[x + \frac{i}{a} \log |\cos ax|\right]$$

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Replacing i by $-i$ -

(ii) $\frac{1}{D+ia} \cdot \sec ax = e^{-iax} \int e^{iax} \cdot \sec ax \, dx$

$= e^{-iax} \int (\cos ax + i \sin ax) \sec ax \, dx$

$= e^{-iax} \int \left(1 + i \frac{\sin ax}{\cos ax}\right) dx$

$= e^{-iax} \int \left(1 - \frac{i}{a} \cdot \frac{(-a \sin ax)}{\cos ax}\right) dx$

$= e^{-iax} \left[x - \frac{i}{a} \log |\cos ax|\right]$

$P.I = \frac{1}{2ia} \left[\left\{e^{iax}\left(x + \frac{i}{a} \log|\cos ax|\right)\right\} - \left\{e^{-iax}\left(x - \frac{i}{a} \log|\cos ax|\right)\right\}\right]$

$= \frac{1}{2ia} \left[x (e^{iax} - e^{-iax}) + \frac{i}{a} \log|\cos ax| (e^{iax} + e^{-iax})\right]$

$= \frac{1}{2ia} \left[x (2i \sin ax) + \frac{i}{a} \log|\cos ax| \cdot 2 \cos ax\right]$

$C.S = C.F + P.I$

$y(x) = (c_1 \cos ax + c_2 \sin ax) + \frac{1}{2ia} \left[2xi \sin ax + \frac{2i}{a} \cos ax \log|\cos ax|\right]$

$$y(x) = (c_1 \cos ax + c_2 \sin ax) + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \cdot \log|\cos ax|$$

Que (3) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$

Que (4) $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x + 5^{\log x}$

Soln (2) Given that -

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x \quad \text{--- (1)}$$

Let $x = e^z$

$\Rightarrow z = \log x$

Then, $x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Using in eqn (1) -

$[D(D-1)y - Dy - 3y] = e^{2z}.z$

$D^2y - Dy - Dy - 3y = e^{2z}.z$

$[D^2 - 2D - 3]y = z.e^{2z} \quad$ — (3)

Now, A.E ~~is~~ eqn of (3)

$m^2 - 2m - 3 = z.e^{2z}$

$(m-3)(m+1) = 0$

$m = 3, -1$

$$\boxed{C.F = c_1e^{-z} + c_2e^{3z}}$$

$$P.I = \frac{1}{D^2-2D-3} z.e^{2z}$$

$$= e^{2z}. \frac{1}{(D+2)^2 - 2(D+2) - 3} z$$

$$= e^{2z} \frac{1}{D^2 - 2D - 3} z$$

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$$\frac{e^{2z}}{-3} \frac{1}{\left[1-\frac{2D}{3}-\frac{D^2}{3}\right]}$$

$$= \frac{e^{2z}}{-3}\left[1-\frac{2D}{3}-\frac{D^2}{3}\right]^{-1} z$$

$$= \frac{e^{2z}}{-3}\left[1+\frac{2D}{3}+\frac{D^2}{3}+ \text{--}\right] z$$

$$= \frac{e^{2z}}{-3}\left[z+\frac{2}{3}\right].$$

Complete solution is -

$$y = C.F + P.I$$

$$y = c_1 e^{-z} + c_2 e^{3z} - \frac{e^{2z}}{-3}\left[z+\frac{2}{3}\right]$$

$$\boxed{y = \frac{c_1}{x} + c_2 x^3 - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right)}$$

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Ques $\frac{d^2y}{dt^2} + x = \cos t$

$\frac{d^2x}{dt^2} + y = \sin t$.

Sol^n. Given that -

$$\frac{d^2x}{dt^2} + y = \sin t \quad -①$$

$$\frac{d^2y}{dt^2} + x = \cos t$$

then, $D^2x + y = \sin t \quad -①$

$D^2y + x = \cos t \quad -②$

multi $\cdot D^2$ by ① & sub ② from ① -

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$(D^4x + D^2y) - (D^2y + x) = D^2\sin t - \cos t$

$(D^4 - 1)x = -\sin t - \cos t$

Now, A.E. is -

$m^4 - 1 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow (m+1)(m-1)(m^2+1) = 0$

$\Rightarrow m = -1, 1, i, -i$

Now, $C.F = c_1e^{-t} + c_2e^{t} + (c_3\cos t + c_4\sin t)$

$P.I = \frac{1}{D^4 - 1}(-\sin t - \cos t)$

$= (-1)\left[\frac{1}{D^2 \cdot D^2 - 1}\sin t + \frac{1}{D^2 \cdot D^2 - 1}\cos t\right]$

$= (-1)\left[\frac{1}{(-1)(-1)-1}\sin t + \frac{1}{(-1)(-1)-1}\cos t\right]$ {case fail}

$= (-1)\left[\frac{t}{4D^2}\left\{\int(\sin t + \cos t)\,dt\right\}\right]$

$= (-1)\left[\frac{t}{4D^2}\left\{\int(-\cos t + \sin t)\,dt\right\}\right]$

$= -1\left[\frac{t}{4}\left\{\int(-\sin t - \cos t)\,dt\right\}\right]$

$= \left[\frac{t}{4}(\cos t - \sin t)\right]$

$C.S = C.F + P.I$

$x(t) = c_1e^{-t} + c_2e^{t} + (c_3\cos t + c_4\sin t) - \frac{t}{4}(\cos t - \sin t)$

From eq^n ②, -

Que. (1) Solve the differential equation by using normal Form.

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$$

Soln. Given that -

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\cdot\sin 2x \quad -(1)$$

Compare with -

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$P = -4x$, $Q = 4x^2-1$ & $R = -3e^{x^2}\sin 2x$.

Taking,

$$u = e^{-1/2\int P dx}$$

$$u = e^{-1/2\int(-4x)dx} = e^{x^2}$$

Let $\boxed{y = u\cdot v}$ be the cs of eqn (1)

Then, $\frac{d^2v}{dx^2} + Iv = \frac{R}{v}$

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Where, $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$= (4x^2-1) - \frac{1}{4}(-4x)^2 - \frac{1}{4}(-4)$

$= 4x^2 - 1 - 4x^2 + 2$

$\boxed{I = 1}$

$\frac{d^2v}{dx^2} + v = \frac{-3e^{x^2}\sin 2x}{e^{x^2}}$

$\frac{d^2v}{dx^2} + v = -3\sin 2x$ —②

Now, A.E of ② is –

$m^2+1=0$

$m = \pm i$

$\boxed{C.F = c_1\cos x + c_2\sin x}$

$P.I = \frac{1}{(D^2+1)}(-3\sin 2x)$

$= \frac{1}{-4+1}(-3\sin 2x)$

$P.I = \sin 2x$

$C.S = C.F + P.I$

$v(x) = c_1\cos x + c_2\sin x + \sin 2x$ be the solution of eq^n ②.

Then solution of eq^n ① is given by –

$y = u.v$

$\boxed{y = e^{x^2}[c_1\cos x + c_2\sin x + \sin 2x]}$

Que② $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2-8)y = x^2 \cdot e^{-x^2/2}$ (2012,13).

Sol^n:- Given that –

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$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2-8)y = x^2 \cdot e^{-x^2/2} \quad \text{—①}$$

Now, compare with :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q \cdot y = R$$

$P = 2x$, $Q = x^2 - 8$ & $R = x^2$

Taking,

$$u = e^{-1/2\int P dx}$$

$$= e^{-1/2\int 2x dx} = e^{-x^2/2}$$

Let $y = u.v$ be the c.s of eqⁿ ①

Then, $\frac{d^2v}{dx^2} + Iv = \frac{R}{u}$

where, $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$$= (x^2-8) - \frac{1}{4}(2x)^2 - \frac{1}{2} \times 2$$

$$= x^2 - 8 - x^2 - 1$$

$$\boxed{I = -9}$$

$$\frac{d^2v}{dx^2} - 9v = \frac{x^2 \cdot e^{-x^2/2}}{e^{-x^2/2}}$$

$$\Rightarrow \boxed{\frac{d^2v}{dx^2} - 9v = x^2}$$

Now, A.E is -

$$m^2 - 9 = 0$$

$$m = \pm 3$$

$$\boxed{C.F = C_1e^{-3x} + C_2e^{3x}}$$

$$P.I = \frac{1}{D^2-9} \cdot x^2$$

$$= \frac{-1}{9\left[1 - \frac{D^2}{9}\right]} \cdot x^2$$

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$$= -\frac{1}{9}\left[1-\frac{D^2}{9}\right]^{-1}\cdot x^2$$

$$= -\frac{1}{9}\left[1+\frac{D^2+D^4}{9}+\cdots\right]\cdot x^2$$

$$= -\frac{1}{9}\left[x^2+\frac{2}{9}+0\right]$$

C.S = C.F + P.I

$\Rightarrow v(x) = c_1 e^{-3x} + c_2 e^{3x} - \frac{1}{9}\left(x^2 + 2/9\right)$

Then, solution of eq^n (1) is given by:

$y = u.v$

$$\boxed{y = e^{-x^2/2}\left[c_1 e^{-3x} + c_2 e^{3x} - \frac{1}{9}(x^2+2/9)\right]}$$

Ans

Solution By changing Independent Variable:

Consider the equation -

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Sqn ① Given that -

$$\frac{dx}{dt} + 2x + 4y = 1 + 4t$$

$$\frac{dy}{dt} + x - y = \frac{3t^2}{2}$$

Let $\frac{d}{dt} = D$

Then, $(D+4)x + 4y = 1 + 4t$ — ①

$x + (D-1)y = \frac{3}{2}t^2$ — ②

Multiplying eqn ② by $(D+4)$,

$(D+4)x + (D-1)(D+4)y = \frac{3}{2}t^2$ ---- ③

Now, Subtracting eqn ① from ③ -

$[(D-1)(D+4) - 4]y = \frac{3}{2}(D+4)t^2 - 1 - 4t$

$(D^2 + 3D - 8)y = \frac{3}{2}(2t + 4t^2) - 1 - 4t$

$(D^2 + 3D - 8)y = 3t + 6t^2 - 1 - 4t$

$(D^2 + 3D - 8)y = 6t^2 - t - 1$

Now, A.E is -

$m^2 + 3m - 8 = 0$

$$m = \frac{-3 \pm \sqrt{9 - 4(-8)}}{2}$$

$$m = \frac{-3 \pm \sqrt{41}}{2}$$

$$C.F = e^{-3/2t}\left[c_1 \cosh\frac{\sqrt{41}}{2}t + c_2 \sinh\frac{\sqrt{41}}{2}t\right]$$

$$P.I = \frac{1}{D^2 + 3D - 8}(6t^2 - t - 1)$$

$$= \frac{1}{-8\left(1 - \frac{D^2}{8} - \frac{3}{8}D\right)}(6t^2 - t - 1)$$

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$$= -\frac{1}{8}\left[1-\left(\frac{D^2}{8}+\frac{3}{8}D\right)\right]^{-1}(6t^2-t-1)$$

$$= -\frac{1}{8}\left[1+\frac{D^2}{8}+\frac{3}{8}D+\cdots\right](6t^2-t-1)$$

$$= -\frac{1}{8}\left[6t^2-t-1+\frac{12}{8}+\frac{36}{8}t-\frac{3}{8}\right]$$

$$= -\frac{1}{8}\left[6t^2+\frac{7}{2}t+\frac{1}{8}\right]$$

$$\therefore\ y = e^{-3/2t}\left[c_1\cosh\frac{\sqrt{41}}{2}+c_2\sinh\frac{\sqrt{41}t}{2}\right]-\frac{1}{8}\left(6t^2+\frac{7}{2}t+\frac{1}{8}\right)$$

Now, $\frac{dy}{dt} = e^{-3/2t}\left[\frac{\sqrt{41}}{2}c_1\sinh t+\frac{\sqrt{41}}{2}c_2\cosh\frac{\sqrt{41}}{2}\right]+$

$$\left[c_1\cosh\frac{\sqrt{41}}{2}t+c_2\sinh\frac{\sqrt{41}t}{2}\right]\left(-\frac{3}{2}e^{-3/2t}\right)-\frac{1}{8}\left(12t+\frac{7}{2}\right)$$

$$= e^{-3/2t}\left[c_1\left(\frac{\sqrt{41}}{2}-\frac{3}{2}\right)\cos\frac{\sqrt{41}}{2}t+c_2\left(\frac{\sqrt{41}}{2}-\frac{3}{2}\right)\sinh\frac{\sqrt{41}t}{2}\right]$$

$$-\frac{1}{8}\left(12t+\frac{7}{2}\right).$$

Putting these value in y in eqn ② -

$$x = \frac{3}{2}t^2+e^{-3/2t}\left[c_1\cosh\frac{\sqrt{41}t}{2}+c_2\sinh\frac{\sqrt{41}t}{2}\right]-\frac{1}{8}\left(6t^2+\frac{7}{2}t+\frac{1}{8}\right)$$

$$-e^{-3/2t}\left[c_1\left(\frac{\sqrt{41}}{2}-\frac{3}{2}\right)\cosh\frac{\sqrt{41}}{2}t+c_2\left(\frac{\sqrt{41}}{2}-\frac{3}{2}\right)\sinh\frac{\sqrt{41}t}{2}\right]+\frac{1}{8}$$

$$\left(12t+\frac{7}{2}\right)$$

$$= e^{-3/2t}\left[c_1\left(1-\frac{\sqrt{41}}{2}+\frac{3}{2}\right)\cosh\frac{\sqrt{41}}{2}t+c_2\left(1-\frac{\sqrt{41}}{2}+\frac{3}{2}\right)\sinh\frac{\sqrt{41}}{2}t\right]$$

$$+\frac{1}{8}\left(6t^2+\frac{17}{2}t+3\right)$$

Hence, Required Solution is -

$$x = e^{-3/2t}\left[c_1\left(\frac{5}{2}-\frac{\sqrt{41}}{2}\right)\cosh\frac{\sqrt{41}}{2}t+c_2\left(\frac{5}{2}-\frac{\sqrt{41}}{2}\right)\sinh\frac{\sqrt{41}t}{2}\right]$$
$$+\frac{1}{8}\left[6t^2+\frac{17}{2}t+3\right]$$

Ques ① Solve by changing the Independent Variable.

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$$

Sol^n:- Given that :-

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2 \quad \text{---} ①$$

compare with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$.

then, $P = \frac{-1}{x}$, $Q = -4x^2$ & $R = 8x^2\sin x^2$

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Let $z = f(x)$

then eq^n (I) becomes:-

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

Let $Q_1 = -4$:

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \Rightarrow Q_1 \left(\frac{dz}{dx}\right)^2 = Q$$

$$\Rightarrow (-4)\left(\frac{dz}{dx}\right)^2 = -4x^2$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = x^2$$

$$\Rightarrow \frac{dz}{dx} = x$$

$$\Rightarrow dz = x\,dx$$

on integration -

$$\boxed{z = x^2/2}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{1 + \left(-\frac{1}{x}\right)(x)}{x^2} = 0$$

$Q = -4$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{8x^2 \sin x^2}{x^2} = 8 \sin x^2.$$

$$\frac{d^2y}{dz^2} + 0 \times \frac{dy}{dz} + (-4)y = 8\sin(2z)$$

$$\Rightarrow \frac{d^2y}{dz^2} - 4y = 8\sin(2z)$$

Now, A.E is -

$m^2 - 4 = 0$

$\therefore m = \pm 2$

$$C.F = c_1 e^{-2z} + c_2 e^{2z}$$

$$P.I = \frac{1}{D^2 - 4} 8\sin(2z)$$

$= e^{8x} \frac{1}{(-8)} \sin(2z)$

$= -\sin(2z)$

C.S = C.F + P.I

$y(z) = c_1 e^{-2z} + c_2 e^{2z} - \sin(2z)$

$\because z = \frac{x^2}{2}$

$y(x) = c_1 e^{-2\left(\frac{x^2}{2}\right)} + c_2 e^{2\left(\frac{x^2}{2}\right)} - \sin 2\left(\frac{x^2}{2}\right)$

$$\boxed{y(x) = c_1 e^{-x^2} + c_2 e^{x^2} - \sin x^2}$$

Q② Solve by changing the Independent variable.

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$$

Sol. Given that -

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4 \quad \text{—①}$$

Compare with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —①

Here, $P = \frac{-1}{x}$, $Q = 4x^2$ & $R = x^4$

Let $z = f(x)$

then eqn ① becomes -

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + Q_1y = R_1$$

Let $Q_1 = 4$

$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \Rightarrow Q_1\left(\frac{dz}{dx}\right)^2 = Q.$

$\Rightarrow 4\left(\frac{dz}{dx}\right)^2 = 4x^2$

$\Rightarrow \frac{dz}{dx} = x^2$

$\Rightarrow dz = x\,dx$

On Integration -

$$z = \frac{x^2}{2}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{1 + \left(-\frac{1}{x}\right)(x)}{x^2}$$

$Q_1 = 4$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{x^2} \Rightarrow x^2$$

$$\frac{d^2y}{dx^2} + 0 \times \frac{dy}{dx} + (4)y = x^2$$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = x^2\ 2z$$

Now, A.E is -

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$C.F = c_1 \cos 2z + c_2 \sin 2z$$

$$P.I = \frac{1}{D^2+4} \cdot \sin 2z$$

$$= \frac{1}{4}\left[1 + \frac{D^2}{4}\right]^{-1} (2z)$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} + \cdots\right] \cdot (2z)$$

$$= \frac{1}{4}[2z - 0] = \frac{z}{2}$$

$C.S = C.F + P.I$

$$y(z) = (c_1 \cos 2z + c_2 \sin 2z) + \frac{z}{2}$$

$$\because z = \frac{x^2}{2}$$

$$y(x) = c_1 \cos\left\{2\left(\frac{x^2}{2}\right)\right\} + c_2 \sin\left\{2\left(\frac{x^2}{2}\right)\right\} + \frac{1}{2}\left(\frac{x^2}{2}\right)$$

$$\boxed{y(x) = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}}$$ Ans

Imp** Que.

Solve the given differential Equation by using Variation of Parameter –

$y'' + y = \operatorname{cosec} x$

Soln.- Given that :-

$y'' + y = \operatorname{cosec} x$

$(D^2 + 1)\, y = \operatorname{cosec} x$

Now, A.E is –

$m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$C.F = c_1 \cos x + c_2 \sin x$

Then $\Rightarrow y_1 = \cos x$ & $y_2 = \sin x$

$P.I = u \cos x + v \sin x$

Where, $u = \int \frac{-y_2 R}{W} dx$

$R = \operatorname{cosec} x$, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1$

$u = \int \frac{(-\sin x) \cdot \operatorname{cosec} x}{1} dx = -x$ ans

$v = \int \frac{y_1 R}{W}$

$= \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx$

$$= \int \frac{\cos x}{\sin x}$$

$$v = \log |\sin x|$$

Now,

$$P.I = -x\cos x + \sin \log|\sin x|$$

Now, general solⁿ of eqⁿ ① -

$$G.S = C.F + P.I$$

$$\boxed{y(x) = c_1\cos x + c_2\sin x - x\cos x + \sin \log|\sin x|}$$ Ans

Que②: $y'' - 2y' = e^x \sin x$

Solⁿ Given that -

$$y'' - 2y' = e^x \sin x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$$

$$D^2y - 2Dy = e^x \sin x$$

$$(D^2 - 2D)y = e^x \sin x$$

Now, A.E is -

$$m^2 - 2m = 0$$

$$\Rightarrow \boxed{m = 0, 2}$$

$$C.F = c_1 e^{0x} + c_2 e^{2x}$$

$$\boxed{C.F = c_1 + c_2 e^{2x}}$$

Now, $y_1 = 1$ & $y_2 = e^{2x}$

$$P.I = u + v e^{2x}$$

where, $u = \int \frac{-y_2 R}{W} dx$

$R = e^x \sin x$; $W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix}$

$$W = 2e^{2x} - 0$$

$$\boxed{W = 2e^{2x}}$$

$$u = \int \frac{-e^{2x} \cdot e^x \sin x \, dx}{2e^{2x}} dx = \frac{-1}{2}\int e^x \cdot \sin x \, dx$$

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$\because \int e^{ax}\sin nx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\therefore \int e^{ax}\cos nx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin ax)$.

Now, $u = \frac{-1}{2}\int e^{x}.\sin x\,dx$

$= \frac{-1}{2}\left[\frac{e^{x}}{1^2+1^2}(\sin x - \cos x)\right]$

$u = \frac{-e^{x}}{4}(\sin x - \cos x)$.

$v = \int \frac{e^{x}.\sin x}{2e^{2x}}dx$

$= \frac{1}{2}\int e^{-x}\sin x\,dx$

$= \frac{1}{2}\left[\frac{e^{-x}}{2}(-\sin x - \cos x)\right]$

$v = \frac{-e^{-x}}{4}(\sin x + \cos x)$

Then, $\therefore P.I = \left(\frac{-e^{x}}{4}(\sin x - \cos x) + \left\{\frac{e^{-x}}{4}(\sin x + \cos x)\right\}e^{2x}\right.$

$C.S = C.F + P.I$

$y_1(x) = c_1 + c_2e^{2x} + \left(\frac{-e^{x}}{4}\right)(\sin x - \cos x) + \left\{\frac{-e^{-x}}{4}(\sin x + \cos x)\right\}e^{2x}$

$y(x) = c_1 + c_2e^{2x} - \frac{1}{2}e^{x}\sin x$

Scanned by CamScanner

Que: A body executes damped forced Vibration given in the eqn $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt}\sin wt$ solve, the eqn for both the case $w^2 \neq b^2 - k^2$ & when $w^2 = b^2 - k^2$

Soln: Given that -

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt}\sin wt$$

$$(D^2 + 2kD + b^2)x = e^{-kt}\sin wt \quad -(1)$$

Now, A.E is -

$$m^2 + 2km + b^2 = 0$$

$$m = \frac{-2k \pm \sqrt{(2k)^2 - 4 \times b^2 \times 1}}{2 \times 1}$$

$$= \frac{-2k \pm \sqrt{4k^2 - 4b^2}}{2}$$

$$= \frac{-2k \pm 2\sqrt{k^2 - b^2}}{2}$$

$$= -k \pm \sqrt{-(b^2 - k^2)}$$

$$m = -k \pm i\sqrt{b^2 - k^2}$$

$$C.F = [c_1 \cos(\sqrt{b^2 - k^2})t + \sin(\sqrt{b^2 - k^2})t]e^{-kt}$$

Now,

$$P.I = \frac{1}{D^2 + 2kD + b^2} e^{-kt}\sin wt$$

$$= e^{-kt} \frac{1}{(D-k)^2 + 2k(D-k) + b^2} \sin wt$$

$$= e^{kt} \frac{1}{D^2 - 2kD + k^2 + 2kD - 2k^2 + b^2} \sin wt$$

$$= e^{-kt} \frac{1}{D^2 + (b^2 - k^2)} \sin wt$$

Date: / /

$$= \frac{e^{-kt} \sin\omega t}{-\omega^2+(b^2-k^2)} \quad ; \quad \omega^2 \neq b^2-k^2$$

If $\omega^2 = b^2-k^2$

Then, $P.I = e^{-kt} \frac{t}{2D} \sin\omega t$

$$= \frac{te^{-kt}}{2} \int \sin\omega t \, dt$$

$$= \frac{-te^{-kt}}{2} \frac{\cos\omega t}{\omega}$$

$$= \frac{-te^{-kt} \cos\omega t}{2\omega}$$

Now soln of eqn ① is given by -

Case①:- $\omega^2 \neq b^2-k^2$

$$x(t) = [c_1 \cos(\sqrt{b^2-k^2})t + c_2 \sin[\sqrt{b^2-k^2}]t]e^{-kt} + \frac{e^{-kt} \sin\omega t}{-\omega^2+(b^2-k^2)}$$

Case②:- If $\omega^2 = b^2-k^2$

$$x(t) = c_1 \cos(\sqrt{b^2-k^2})t + c_2 \sin(\sqrt{b^2-k^2})t - \frac{te^{-kt} \cos\omega t}{2\omega}$$

BUDDHA SERIES

College - BUDDHA INSTITUTE OF TECHNOLOGY
(AKTU CODE-525)

DEPARTMENT OF APPLIED SCIENCE & HUMANITIES-I

Subject: Engineering Mathematics-II
(BAS203)

Unit-II

Faculty Name : Dr. Rajan Kumar Dubey

Buddha Series (Unit-02)
Engg. Mathematics-II

Dr. Rajan Kumar Dubey

Q1. Find the Laplace Transform of the following function:

$$\int_0^\infty \frac{e^{-t}\sin 2t}{t}\,dt$$

(AKTU 2024-25)

Solution:
We use the standard result:

$$\int_0^\infty \frac{e^{-at}\sin bt}{t}\,dt = \tan^{-1}\left(\frac{b}{a}\right)$$

Here, $a = 1$ and $b = 2$.

$$\Rightarrow \int_0^\infty \frac{e^{-t}\sin 2t}{t}\,dt = \tan^{-1}(2)$$

$$\boxed{\tan^{-1}(2)}$$

Laplace Transform Watch Now Lecture -01 (Click Here)

Laplace Transform Watch Now Lecture -02 (Click Here)

Q2. Find the Laplace Transform of the following function:

$$F(t) = \frac{1 - \cos t}{t^2}$$

(AKTU 2023-24)

Solution:
We use the standard Laplace Transform:

$$\mathcal{L}\left\{\frac{1 - \cos at}{t^2}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$$

Here, $a = 1$.

$$\Rightarrow \mathcal{L}\left\{\frac{1 - \cos t}{t^2}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$$

$$\boxed{\tan^{-1}\left(\frac{1}{s}\right)}$$

Laplace Transform Watch Now Lecture -01 (Click Here)

Laplace Transform Watch Now Lecture -02 (Click Here)

Q3. Find the Laplace Transform of the following function:

$$F(t) = e^{-2t}\sin^3 t$$

Solution:
First use identity:

$$\sin^3 t = \frac{3\sin t - \sin 3t}{4}$$

So,

$$F(t) = e^{-2t}\cdot\frac{3\sin t - \sin 3t}{4}$$

$$= \frac{3}{4}e^{-2t}\sin t - \frac{1}{4}e^{-2t}\sin 3t$$

Now using:

$$\mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$$

$$\mathcal{L}\{F(t)\} = \frac{3}{4}\cdot\frac{1}{(s+2)^2+1} - \frac{1}{4}\cdot\frac{3}{(s+2)^2+9}$$

$$\boxed{\frac{3}{4[(s+2)^2+1]} - \frac{3}{4[(s+2)^2+9]}}$$

Laplace Transform Watch Now Lecture -01 (Click Here)

Laplace Transform Watch Now Lecture -02 (Click Here)

Q4 (a). Find Laplace Transform of $f(t) = \sin 2t \cos 3t$

(AKTU 2024-25)

Solution:
Use identity:

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin 2t \cos 3t = \frac{1}{2}[\sin 5t - \sin t]$$

Taking Laplace Transform:

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left(\frac{5}{s^2+25} - \frac{1}{s^2+1}\right)$$

$$\boxed{\frac{1}{2}\left(\frac{5}{s^2+25} - \frac{1}{s^2+1}\right)}$$

Laplace Transform Watch Now Lecture -01 (Click Here)

Laplace Transform Watch Now Lecture -02 (Click Here)

Q4 (b). Find the Laplace transform of

$$f(t) = t^4 e^{2t}$$

(AKTU 2023-24)

Solution:
We use the formula:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

and shifting property:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

First:

$$\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$$

Now apply shift $s \to s-2$:

$$\mathcal{L}\{t^4 e^{2t}\} = \frac{24}{(s-2)^5}$$

$$\boxed{\frac{24}{(s-2)^5}}$$

Q5. Solve the following differential equation using Laplace Transform:

$$y''' + 2y'' - y' - 2y = 0, \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = 2$$

(AKTU 2024-25)

Solution:

Taking Laplace Transform of both sides:

$$\mathcal{L}\{y'''\} + 2\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

Using formulas:

$$\mathcal{L}\{y'\} = sY - y(0), \quad \mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

Substitute values:

$$(s^3Y - s^2 - 2s - 2) + 2(s^2Y - s - 2) - (sY - 1) - 2Y = 0$$

Simplify:

$$(s^3 + 2s^2 - s - 2)Y = s^2 + 4s + 5$$

$$Y = \frac{s^2 + 4s + 5}{(s^3 + 2s^2 - s - 2)}$$

Factor denominator:

$$s^3 + 2s^2 - s - 2 = (s + 2)(s^2 - 1) = (s + 2)(s - 1)(s + 1)$$

So,

$$Y = \frac{s^2 + 4s + 5}{(s + 2)(s - 1)(s + 1)}$$

Using partial fractions:

$$Y = \frac{A}{s + 2} + \frac{B}{s - 1} + \frac{C}{s + 1}$$

Solving:

$$A = 1, \quad B = 2, \quad C = -1$$

$$Y = \frac{1}{s+2} + \frac{2}{s-1} - \frac{1}{s+1}$$

Taking inverse Laplace:

$$y(t) = e^{-2t} + 2e^{t} - e^{-t}$$

$$\boxed{y(t) = e^{-2t} + 2e^{t} - e^{-t}}$$

Q6. Use convolution theorem to evaluate:

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right]$$

(AKTU 2024-25)

Solution:

We write:

$$\frac{s^2}{(s^2+4)(s^2+9)} = \frac{s}{s^2+4}\cdot\frac{s}{s^2+9}$$

Now,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

So,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t, \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$$

By convolution theorem:

$$f(t) = \int_0^t \cos 2u \cdot \cos 3(t-u)\, du$$

Using identity and simplifying:

$$f(t) = \frac{1}{5}(\cos 2t - \cos 3t)$$

$$\boxed{\frac{1}{5}(\cos 2t - \cos 3t)}$$

Q7. Use convolution theorem to find:

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

(AKTU 2023-24)

Solution:

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

Using convolution:

$$f(t) = \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{a^2}\int_0^t \sin au \sin a(t-u)\,du$$

Using identity:

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

After simplification:

$$f(t) = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$\boxed{\frac{1}{2a^3}(\sin at - at\cos at)}$$

Q8. Find inverse Laplace of:

$$f(s) = \frac{1}{s^2 + 2s + 2}$$

(AKTU 2022-23)

Solution:
Complete the square:

$$s^2 + 2s + 2 = (s+1)^2 + 1$$

$$f(s) = \frac{1}{(s+1)^2 + 1}$$

Using standard result:

$$\mathcal{L}^{-1}\left(\frac{1}{(s+a)^2 + b^2}\right) = e^{-at}\frac{\sin bt}{b}$$

Here $a = 1,\ b = 1$:

$$f(t) = e^{-t}\sin t$$

$$\boxed{e^{-t}\sin t}$$

Q9. Using Laplace Transform, solve the differential equation:

$$y'' + 4y' + 4y = 6e^{-t}, \quad y(0) = -2,\ y'(0) = 8$$

(AKTU 2022-23)

Solution:

Taking Laplace Transform on both sides:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 6\mathcal{L}\{e^{-t}\}$$

Using formulas:

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}\{y'\} = sY - y(0)$$

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Substitute values $y(0) = -2,\ y'(0) = 8$:

$$(s^2Y + 2s - 8) + 4(sY + 2) + 4Y = \frac{6}{s+1}$$

Simplify:

$$s^2Y + 4sY + 4Y + 2s = \frac{6}{s+1}$$

$$Y(s^2 + 4s + 4) + 2s = \frac{6}{s+1}$$

$$Y(s+2)^2 = \frac{6}{s+1} - 2s$$

$$Y = \frac{6}{(s+1)(s+2)^2} - \frac{2s}{(s+2)^2}$$

Now solve each term separately.

First term:

$$\frac{6}{(s+1)(s+2)^2}$$

Using partial fractions:

$$\frac{6}{(s+1)(s+2)^2} = \frac{6}{s+1} - \frac{6}{s+2} - \frac{6}{(s+2)^2}$$

Taking inverse Laplace:

$$= 6e^{-t} - 6e^{-2t} - 6te^{-2t}$$

Second term:

$$\frac{2s}{(s+2)^2}$$

Write:

$$s = (s+2) - 2$$

$$\frac{2s}{(s+2)^2} = \frac{2}{s+2} - \frac{4}{(s+2)^2}$$

Taking inverse Laplace:

$$= 2e^{-2t} - 4te^{-2t}$$

Combine results:

$$y(t) = (6e^{-t} - 6e^{-2t} - 6te^{-2t}) - (2e^{-2t} - 4te^{-2t})$$

$$y(t) = 6e^{-t} - 8e^{-2t} - 2te^{-2t}$$

$$\boxed{y(t) = 6e^{-t} - 8e^{-2t} - 2te^{-2t}}$$

Laplace Transform Watch Now Lecture -01 (Click Here)

Laplace Transform Watch Now Lecture -02 (Click Here)

BUDDHA SERIES

College - BUDDHA INSTITUTE OF TECHNOLOGY
(AKTU CODE-525)

DEPARTMENT OF APPLIED SCIENCE & HUMANITIES-I

Subject: Engineering Mathematics-II
(BAS203)

Unit-III

Faculty Name : Dr. Rajan Kumar Dubey

Buddha Series (Unit-03)
Engg. Mathematics-II

Dr. Rajan Kumar Dubey

Q1. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^{2n}$$

(AKTU 2024-25)

Solution:
Let

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^{2n}$$

Using Ratio Test,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+1)}{(2n+2)} x^2$$

$$= x^2 \lim_{n\to\infty} \frac{2n+1}{2n+2} = x^2$$

Let

$$L = x^2$$

For convergence,

$$L < 1$$

$$x^2 < 1$$

$$|x| < 1$$

Hence the series is:
Convergent for

$$|x| < 1$$

Divergent for

$$|x| > 1$$

At

$$x = \pm 1$$

the series diverges.

$$\boxed{\text{The series converges for } |x| < 1}$$

Q2. Test for the convergence of the series

$$1 + \frac{x}{2} + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots, \quad x > 0$$

(AKTU 2023-24)

Solution:
General term:

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}x^n$$

Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n+2}x$$

$$= x$$

For convergence,

$$x < 1$$

Hence:
Convergent for

$$0 < x < 1$$

Divergent for

$$x > 1$$

At

$$x = 1$$

the series diverges.

$$\boxed{\text{The series converges for } 0 < x < 1}$$

Q3. Obtain the Fourier series for the function

$$f(x) = x^2, \quad -\pi \leq x \leq \pi$$

(AKTU 2024-25)

Solution:
Since

$$f(x) = x^2$$

is an even function, therefore

$$b_n = 0$$

Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Constant term:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2 dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2\pi^2}{3}$$

$$\frac{a_0}{2} = \frac{\pi^2}{3}$$

Coefficient:

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

After integration by parts,

$$a_n = \frac{4(-1)^n}{n^2}$$

Hence Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$\boxed{x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx}$$

Putting

$$x = \pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Q4. Discuss the convergence of the sequence

$$u_n = \sin\left(\frac{1}{n}\right)$$

(AKTU 2022-23)

Solution:
Given sequence:

$$u_n = \sin\left(\frac{1}{n}\right)$$

As

$$n \to \infty$$

$$\frac{1}{n} \to 0$$

Therefore,

$$u_n = \sin\left(\frac{1}{n}\right) \to \sin 0$$

$$= 0$$

Hence

$$\lim_{n\to\infty} u_n = 0$$

Therefore the sequence is convergent and its limit is

$$\boxed{0}$$

$$\boxed{\text{The sequence is convergent}}$$

Q5. Test the convergence of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{x}{4\cdot 5\cdot 6} + \frac{x^2}{7\cdot 8\cdot 9} + \cdots$$

(AKTU 2024-25)

Solution:

The general term is

$$u_n = \frac{x^{n-1}}{(3n-2)(3n-1)(3n)}$$

Using Ratio Test,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n\to\infty} |x| \frac{(3n-2)(3n-1)(3n)}{(3n+1)(3n+2)(3n+3)}$$

$$= |x|$$

For convergence,

$$|x| < 1$$

Hence the series is convergent for

$$|x| < 1$$

and divergent for

$$|x| > 1$$

At

$$|x| = 1$$

the terms behave like

$$\frac{1}{n^3}$$

which is convergent.

Hence,

$$\boxed{\text{The series converges for } |x| \leq 1}$$

Q6. Test for the convergence of the series

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots$$

(AKTU 2023-24)

Solution:
The pattern gives

$$u_n = \frac{2^n - 2}{2^n + 1}x^{n-1}, \quad n \geq 1$$

Using Ratio Test,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^{n+1} - 2}{2^{n+1} + 1} \cdot \frac{2^n + 1}{2^n - 2} \right| |x|$$

$$= |x|$$

For convergence,

$$|x| < 1$$

Hence the series converges for

$$|x| < 1$$

and diverges for

$$|x| > 1$$

At

$$x = \pm 1$$

the terms do not tend to zero, therefore divergent.

$$\boxed{\text{The series converges for } |x| < 1}$$

Q7. Find the half range Fourier sine series of

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 4 - x, & 2 < x < 4 \end{cases}$$

(AKTU 2023-24)

Solution:
For half range sine series in

$$0 < x < 4$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{4}$$

where

$$b_n = \frac{2}{4} \int_0^4 f(x) \sin \frac{n\pi x}{4}\, dx$$

$$b_n = \frac{1}{2} \left[\int_0^2 x \sin \frac{n\pi x}{4}\, dx + \int_2^4 (4 - x) \sin \frac{n\pi x}{4}\, dx \right]$$

After integration,

$$b_n = \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Hence the half range sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{4}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{4}}$$

Q8. Discuss the convergence of the sequence

$$a_n = \begin{cases} 1, & \text{if } n = 2^p \text{ for some } p \in N \\ \dfrac{1}{n}, & \text{otherwise} \end{cases}$$

(AKTU 2024-25)

Solution:
For values

$$n = 2^p$$

we have

$$a_n = 1$$

Thus the subsequence

$$a_{2^p} = 1$$

Therefore,

$$\lim_{p \to \infty} a_{2^p} = 1$$

For all other values,

$$a_n = \frac{1}{n}$$

Hence,

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Thus another subsequence tends to

$$0$$

Since two subsequences have different limits,

$$1 \neq 0$$

therefore the sequence has no unique limit.
Hence,

$$\boxed{\text{The sequence is divergent}}$$

BUDDHA SERIES

College - BUDDHA INSTITUTE OF TECHNOLOGY
(AKTU CODE-525)

DEPARTMENT OF APPLIED SCIENCE & HUMANITIES-I

Subject: Engineering Mathematics-II
(BAS203)

Unit-IV

Faculty Name : Dr. Rajan Kumar Dubey

Buddha Series (Unit-04)
Engg. Mathematics-II

Dr. Rajan Kumar Dubey

Q1. Show that the function

$$f(z) = \begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin. **(AKTU 2022-23)**

Solution:

Let

$$f(z) = u + iv$$

Then

$$u(x,y) = \frac{x^4y^5}{x^6+y^{10}}, \qquad v(x,y) = \frac{x^3y^6}{x^6+y^{10}}$$

At origin,

$$u(0,0) = 0, \qquad v(0,0) = 0$$

Now,

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = 0$$

$$\frac{\partial u}{\partial y}(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial x}(0,0) = 0, \qquad \frac{\partial v}{\partial y}(0,0) = 0$$

Hence,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

Therefore Cauchy-Riemann equations are satisfied at the origin.
Now examine derivative:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

Take path

$$y = x^{3/5}$$

Then

$$y^{10} = x^6$$

Hence

$$\frac{f(z)}{z} = \frac{x^3 y^5}{x^6 + y^{10}} = \frac{x^3 x^3}{x^6 + x^6} = \frac{1}{2}$$

But along

$$y = 0$$

$$\frac{f(z)}{z} = 0$$

Since the limit depends on path, derivative does not exist.
Hence

$$\boxed{\text{The function is not analytic at the origin}}$$

Q2. Show that

$$u = \frac{1}{2}\log(x^2 + y^2)$$

is harmonic. Find its harmonic conjugate. **(AKTU 2023-24)**

Solution:

Given

$$u = \frac{1}{2}\log(x^2 + y^2)$$

First derivatives:

$$u_x = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{y}{x^2 + y^2}$$

Second derivatives:

$$u_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Therefore,

$$u_{xx} + u_{yy} = 0$$

Hence

$$\boxed{u \text{ is harmonic}}$$

Using Cauchy-Riemann equations:

$$v_y = u_x = \frac{x}{x^2 + y^2}$$

Integrating w.r.t. y,

$$v = \tan^{-1}\left(\frac{y}{x}\right) + \phi(x)$$

Also,

$$-v_x = u_y = \frac{y}{x^2 + y^2}$$

which gives

$$\phi'(x) = 0$$

Thus,

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

Hence harmonic conjugate is

$$\boxed{v = \tan^{-1}\left(\frac{y}{x}\right)}$$

Q3. Show that

$$f(z) = z|z|$$

is nowhere analytic. **(AKTU 2024-25)**

Solution:

Let

$$z = x + iy$$

Then

$$|z| = \sqrt{x^2 + y^2}$$

Hence

$$f(z) = (x + iy)\sqrt{x^2 + y^2}$$

So,

$$u = x\sqrt{x^2 + y^2}$$

$$v = y\sqrt{x^2 + y^2}$$

Now,

$$u_x = \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$v_y = \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}$$

For Cauchy-Riemann equations,

$$u_x = v_y$$

This gives

$$2x^2 + y^2 = x^2 + 2y^2$$

$$x^2 = y^2$$

which is not true for all points.

Hence Cauchy-Riemann equations are not satisfied anywhere.

Therefore

$$\boxed{f(z) = z|z| \text{ is nowhere analytic}}$$

Q4. Show that

$$u = e^x(x\cos y - y\sin y)$$

is harmonic and find analytic function whose imaginary part is u. **(AKTU 2024-25)**

Solution:

Given

$$u = e^x(x\cos y - y\sin y)$$

Observe that

$$z = x + iy$$

and

$$ze^z = (x+iy)e^x(\cos y + i\sin y)$$

Expanding,

$$ze^z = e^x[(x\cos y - y\sin y) + i(x\sin y + y\cos y)]$$

Therefore,

$$u = e^x(x\cos y - y\sin y)$$

is the real part of

$$ze^z$$

Hence u is harmonic.
So the harmonic conjugate is

$$v = e^x(x\sin y + y\cos y)$$

Therefore analytic function is

$$f(z) = u + iv = ze^z$$

If u is taken as imaginary part, then required analytic function is

$$f(z) = i\,ze^z + C$$

Hence

$$\boxed{f(z) = i\,ze^z + C}$$

Q5. Determine an analytic function $f(z) = u + iv$ whose real part is

$$u(x, y) = e^x(x \cos y - y \sin y)$$

and

$$f(1) = e$$

(AKTU 2022-23)

Solution:
Given

$$u(x, y) = e^x(x \cos y - y \sin y)$$

Observe that

$$z = x + iy$$

and

$$ze^z = (x + iy)e^x(\cos y + i \sin y)$$

Expanding,

$$ze^z = e^x \Big[(x \cos y - y \sin y) + i(x \sin y + y \cos y)\Big]$$

Hence

$$u = e^x(x \cos y - y \sin y)$$

is the real part of

$$ze^z$$

Therefore,

$$f(z) = ze^z + C$$

Now use

$$f(1) = e$$

$$1 \cdot e^1 + C = e$$

$$e + C = e$$

$$C = 0$$

Hence the required analytic function is

$$\boxed{f(z) = ze^z}$$

Q6. Show that

$$u = e^x(x\cos y - y\sin y)$$

is harmonic and find the analytic function for which it is imaginary part. **(AKTU 2023-24)**

Solution:

Given

$$u = e^x(x\cos y - y\sin y)$$

Since

$$ze^z = e^x\Big[(x\cos y - y\sin y) + i(x\sin y + y\cos y)\Big]$$

we note that

$$u = e^x(x\cos y - y\sin y)$$

is harmonic because it is a part of an analytic function.

Now if u is the imaginary part, then required analytic function is

$$f(z) = i\,ze^z + C$$

because multiplying by i interchanges real and imaginary parts.

Hence the required analytic function is

$$\boxed{f(z) = i\,ze^z + C}$$

Q7. Find the bilinear transformation which maps

$$z = 0, -1, i$$

onto

$$w = i, 0, \infty$$

Also find the image of the unit circle $|z| = 1$. **(AKTU 2023-24)**

Solution:

Let

$$w = \frac{az + b}{cz + d}$$

Since

$$z = i \mapsto w = \infty$$

therefore

$$ci + d = 0$$

So

$$d = -ci$$

Hence

$$w = \frac{az + b}{c(z - i)}$$

Since

$$z = -1 \mapsto w = 0$$

we get

$$-a + b = 0$$

$$b = a$$

Thus

$$w = \frac{a(z + 1)}{c(z - i)}$$

Since

$$z = 0 \mapsto w = i$$

$$i = \frac{a}{-ci}$$

$$i = \frac{ai}{c}$$

$$a = c$$

Hence transformation is

$$\boxed{w = \frac{z+1}{z-i}}$$

For image of unit circle

$$|z| = 1$$

Let inverse transformation:

$$z = \frac{iw+1}{w-1}$$

Substitute in

$$|z| = 1$$

$$\left|\frac{iw+1}{w-1}\right| = 1$$

$$|iw+1| = |w-1|$$

This simplifies to

$$\Re(w) = 0$$

Hence image is imaginary axis.

$$\boxed{\text{Image of } |z| = 1 \text{ is the imaginary axis}}$$

Q8. Define analytic function and show that

$$f(z) = z|z|$$

is not analytic anywhere. **(AKTU 2023-24)**

Solution:

A function $f(z)$ is said to be analytic at a point if its derivative exists at that point and in its neighbourhood.

Given

$$f(z) = z|z|$$

Let

$$z = x + iy$$

Then

$$|z| = \sqrt{x^2 + y^2}$$

Therefore

$$f(z) = (x + iy)\sqrt{x^2 + y^2}$$

So,

$$u = x\sqrt{x^2 + y^2}$$

$$v = y\sqrt{x^2 + y^2}$$

Now,

$$u_x = \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$v_y = \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}$$

For analyticity,

$$u_x = v_y$$

which gives

$$2x^2 + y^2 = x^2 + 2y^2$$

$$x^2 = y^2$$

This is not true for all points.

Hence Cauchy-Riemann equations are not satisfied everywhere.
Therefore,

$$\boxed{f(z) = z|z| \text{ is nowhere analytic}}$$

BUDDHA SERIES

College - BUDDHA INSTITUTE OF TECHNOLOGY
(AKTU CODE-525)

DEPARTMENT OF APPLIED SCIENCE & HUMANITIES-I

Subject: Engineering Mathematics-II
(BAS203)

Unit-V

Faculty Name : Dr. Rajan Kumar Dubey

Buddha Series (Unit-05)
Engg. Mathematics-II

Dr. Rajan Kumar Dubey

Q1. Evaluate the integral using Cauchy Integral Formula

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\, dz$$

where C is the circle

$$|z| = \frac{3}{2}$$

(AKTU 2022-24)

Solution:

The singularities are at

$$z = 0,\ 1,\ 2$$

Inside the circle

$$|z| = \frac{3}{2}$$

the points

$$z = 0,\ 1$$

lie inside and

$$z = 2$$

lies outside.

Using partial fractions,

$$\frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}$$

Hence

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_C \left(\frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}\right) dz$$

By Cauchy Integral Formula,

$$\int_C \frac{1}{z-a}\,dz = 2\pi i$$

for poles inside C.

Therefore

$$= 2(2\pi i) - 1(2\pi i) - 0$$

$$= 2\pi i$$

Hence,

$$\boxed{2\pi i}$$

Q2. Evaluate using Cauchy Integral Formula

$$\oint_C \frac{\sin 2z}{(z+3)(z+1)^2}\,dz$$

where C is rectangle with vertices

$$3 \pm i,\ -2 \pm i$$

(AKTU 2023-24)

Solution:

The singularities are at

$$z = -3,\ -1$$

The rectangle contains

$$z = -1$$

only.

Write

$$\frac{\sin 2z}{(z+3)(z+1)^2} = \frac{\phi(z)}{(z+1)^2}$$

where

$$\phi(z) = \frac{\sin 2z}{z+3}$$

Using Cauchy Integral Formula for derivative,

$$\oint_C \frac{\phi(z)}{(z-a)^2}\,dz = 2\pi i\,\phi'(a)$$

Here

$$a = -1$$

So

$$I = 2\pi i\,\phi'(-1)$$

Now

$$\phi(z) = \frac{\sin 2z}{z+3}$$

Differentiate:

$$\phi'(z) = \frac{2\cos 2z(z+3) - \sin 2z}{(z+3)^2}$$

At

$$z = -1$$

$$\phi'(-1) = \frac{4\cos 2 + \sin 2}{4}$$

Therefore

$$I = 2\pi i \left(\frac{4\cos 2 + \sin 2}{4} \right)$$

$$\boxed{\frac{\pi i}{2}(4\cos 2 + \sin 2)}$$

Q3. Expand

$$f(z) = \frac{7z-3}{z^3 - z^2 - 2z}$$

in the regions:

$$(i)\ 0 < |z| < 1, \qquad (ii)\ 1 < |z| < 2, \qquad (iii)\ 2 < |z| < 3$$

(AKTU 2022-23)

Solution:
Factor denominator:

$$z^3 - z^2 - 2z = z(z-2)(z+1)$$

So

$$\frac{7z-3}{z(z-2)(z+1)} = \frac{3}{2z} + \frac{11}{6(z-2)} - \frac{1}{3(z+1)}$$

Thus

$$f(z) = \frac{3}{2z} + \frac{11}{6(z-2)} - \frac{1}{3(z+1)}$$

(i) For $0 < |z| < 1$

$$\frac{1}{z-2} = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots\right)$$

$$\frac{1}{z+1} = 1 - z + z^2 - z^3 + \cdots$$

Hence

$$f(z) = \frac{3}{2z} - \frac{11}{12}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right) - \frac{1}{3}(1 - z + z^2 - \cdots)$$

(ii) For $1 < |z| < 2$

$$\frac{1}{z+1} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right)$$

$$\frac{1}{z-2} = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)$$

Hence

$$f(z) = \frac{3}{2z} + \frac{11}{6(z-2)} - \frac{1}{3}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right)$$

(iii) For $2 < |z| < 3$

$$\frac{1}{z-2} = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots\right)$$

$$\frac{1}{z+1} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right)$$

Hence

$$f(z) = \frac{3}{2z} + \frac{11}{6}\left(\frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \cdots\right) - \frac{1}{3}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right)$$

Q4. Classify the singularity of

$$f(z) = \frac{e^{1/z}}{z}$$

(AKTU 2022-23)

Solution:
Expand

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

Multiplying by

$$\frac{1}{z}$$

we get

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2!z^3} + \frac{1}{3!z^4} + \cdots$$

The Laurent series contains infinitely many negative powers of z.
Hence the singularity at

$$z = 0$$

is an essential singularity.

$$\boxed{\text{The singularity at } z = 0 \text{ is essential}}$$

Q65. State Cauchy's Integral Theorem and verify it for

$$f(z) = e^{iz}$$

integrated along the boundary of the rectangle with vertices

$$1 - i,\ 1 + i,\ -1 + i,\ -1 - i$$

in counterclockwise direction. **(AKTU 2024-25)**

Solution:

Cauchy's Integral Theorem:

If a function $f(z)$ is analytic inside and on a simple closed contour C, then

$$\oint_C f(z)\, dz = 0$$

Given

$$f(z) = e^{iz}$$

Since exponential function is analytic everywhere, therefore

$$e^{iz}$$

is analytic inside and on the rectangle.

Hence by Cauchy's Integral Theorem,

$$\oint_C e^{iz}\, dz = 0$$

Therefore,

$$\boxed{\oint_C e^{iz}\, dz = 0}$$

Hence Cauchy's theorem is verified.

Q66. Evaluate

$$\int_C \frac{e^z}{(z-1)(z-4)}\,dz$$

where C is the circle

$$|z| = 2$$

using Cauchy's integral formula. **(AKTU 2023-24)**

Solution:

The singularities are at

$$z = 1,\ 4$$

Inside the circle

$$|z| = 2$$

only

$$z = 1$$

lies inside.

Write

$$\frac{e^z}{(z-1)(z-4)} = \frac{\phi(z)}{z-1}$$

where

$$\phi(z) = \frac{e^z}{z-4}$$

By Cauchy's Integral Formula,

$$\oint_C \frac{\phi(z)}{z-a}\,dz = 2\pi i\,\phi(a)$$

Here

$$a = 1$$

Therefore

$$I = 2\pi i\,\phi(1)$$

$$= 2\pi i\left(\frac{e}{1-4}\right)$$

$$= -\frac{2\pi i e}{3}$$

Hence,

$$\boxed{-\frac{2\pi i e}{3}}$$

Q67. Use contour integration to evaluate

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$$

(AKTU 2023-24)

Solution:
Put

$$z = e^{i\theta}$$

Then

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and

$$d\theta = \frac{dz}{iz}$$

Also,

$$\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$$

Substituting,

$$I = \oint_C \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{5 + 2\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

After simplification,

$$I = \frac{1}{2i} \oint_C \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)}\, dz$$

Factor:

$$2z^2 + 5z + 2 = (2z + 1)(z + 2)$$

Poles inside $|z| = 1$ are

$$z = 0, \quad z = -\frac{1}{2}$$

Sum of residues gives

$$\text{Res}_{z=0} + \text{Res}_{z=-1/2} = \frac{1}{3}i$$

Therefore

$$I = 2\pi i \left(\frac{1}{3}i\right)$$

$$= \frac{2\pi}{3}$$

Hence,

$$\boxed{\frac{2\pi}{3}}$$

Q68. Evaluate using contour integration

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$$

where C is the circle

$$|z| = 2$$

(AKTU 2024-25)

Solution:
The singularities are at

$$z = 1, \quad z = -\frac{3}{2}$$

Both lie inside the circle.
For pole at

$$z = 1$$

write

$$\frac{12z-7}{(z-1)^2(2z+3)} = \frac{\phi(z)}{(z-1)^2}$$

where

$$\phi(z) = \frac{12z-7}{2z+3}$$

Using Cauchy's formula for derivative,

$$\oint_C \frac{\phi(z)}{(z-a)^2}\,dz = 2\pi i\,\phi'(a)$$

Now

$$\phi'(z) = \frac{12(2z+3) - 2(12z-7)}{(2z+3)^2}$$

$$= \frac{50}{(2z+3)^2}$$

At

$$z = 1$$

$$\phi'(1) = \frac{50}{25} = 2$$

Contribution from $z = 1$:

$$2\pi i(2) = 4\pi i$$

For simple pole at

$$z = -\frac{3}{2}$$

Residue:

$$\text{Res} = \lim_{z \to -3/2} \frac{12z - 7}{2(z-1)^2}$$

$$= \frac{-18 - 7}{2\left(-\frac{5}{2}\right)^2}$$

$$= -2$$

Contribution:

$$2\pi i(-2) = -4\pi i$$

Total:

$$4\pi i - 4\pi i = 0$$

Hence,

$$\boxed{0}$$

Q69. Expand

$$f(z) = \frac{z}{(z-1)(2-z)}$$

in Laurent series valid for

(a)

$$|z-1| > 1$$

(b)

$$0 < |z-2| < 1$$

(AKTU 2023-24)

Solution:

First,

$$f(z) = \frac{z}{(z-1)(2-z)}$$

Using partial fractions,

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

(a) For

$$|z-1| > 1$$

Let

$$w = z - 1$$

Then

$$2 - z = 1 - w$$

Hence

$$\frac{2}{2-z} = \frac{2}{1-w}$$

For

$$|w| > 1$$

$$\frac{1}{1-w} = -\frac{1}{w}\left(1 + \frac{1}{w} + \frac{1}{w^2} + \cdots\right)$$

Thus

$$f(z) = \frac{1}{w} - \frac{2}{w}\left(1 + \frac{1}{w} + \frac{1}{w^2} + \cdots\right)$$

$$\boxed{f(z) = -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots}$$

(b) For

$$0 < |z-2| < 1$$

Let

$$w = z - 2$$

Then

$$z = w + 2$$

$$z - 1 = w + 1$$

So

$$f(z) = \frac{w+2}{(w+1)(-w)}$$

$$= -\frac{1}{w} + \frac{1}{w+1}$$

Now

$$\frac{1}{w+1} = 1 - w + w^2 - w^3 + \cdots$$

Hence

$$f(z) = -\frac{1}{w} + 1 - w + w^2 - \cdots$$

$$\boxed{f(z) = -\frac{1}{z-2} + 1 - (z-2) + (z-2)^2 - \cdots}$$